The Curl of a Vector Field

Say $\nabla \times \mathbf{A}(\bar{r}) = \mathbf{B}(\bar{r})$. The **mathematical** definition of Curl is given as:

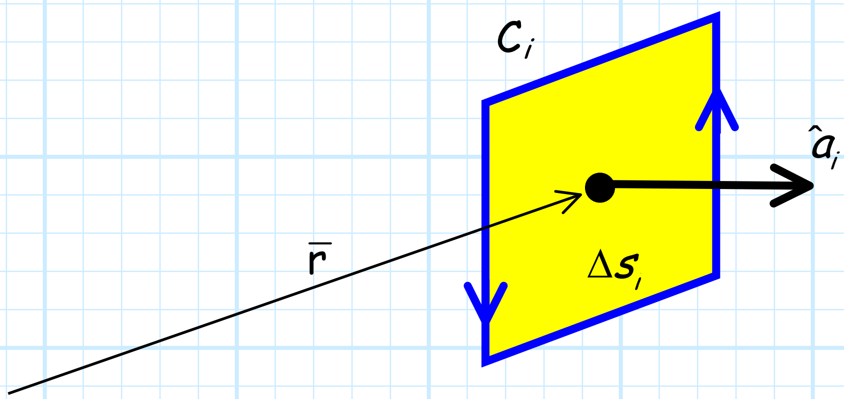
$$B_i(\bar{r}) = \lim_{\Delta s \to 0} \frac{\oint_{C_i} \mathbf{A}(\bar{r}) \cdot \overline{d\ell}}{\Delta s_i}$$

This rather complex equation requires some **explanation** !

* $B_i(\bar{r})$ is the scalar component of vector $\mathbf{B}(\bar{r})$ in the direction defined by unit vector $\hat{a}_i$ (e.g., $\hat{a}_x, \hat{a}_\rho, \hat{a}_\theta$).

* The small surface Δs_i is centered at point $\bar{r}$, and oriented such that it is normal to unit vector $\hat{a}_i$.

* The contour C_i is the closed contour that surrounds surface Δs_i.



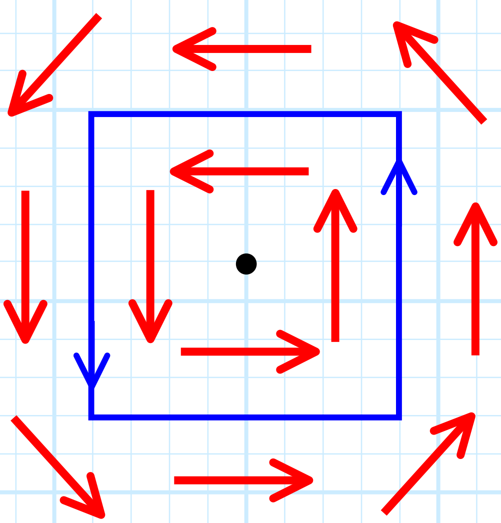
Note that this derivation must be completed for **each** of the **three** orthonormal base vectors in order to completely define $\mathbf{B}(\bar{r}) = \nabla \times \mathbf{A}(\bar{r})$.

Q: *What does curl tell us ?*

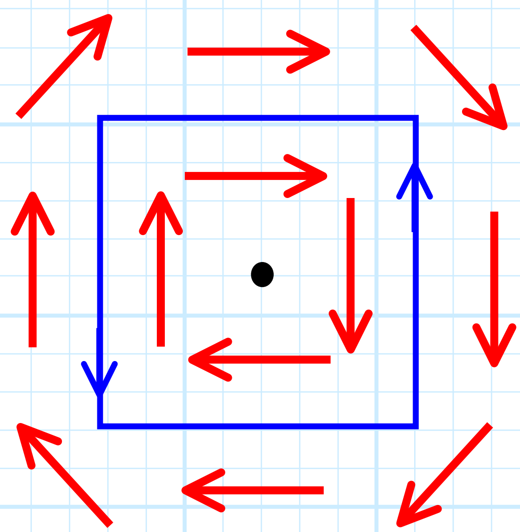
A: Curl is a measurement of the **circulation** of vector field $\mathbf{A}(\bar{r})$ around point $\bar{r}$.

If a component of vector field $\mathbf{A}(\bar{r})$ is pointing in the direction $\overline{d\ell}$ at every point on contour C_i (i.e., **tangential** to the contour). Then the line integral, and thus the curl, will be **positive**.

If, however, a component of vector field $\mathbf{A}(\bar{r})$ points in the opposite direction ($-\overline{d\ell}$) at every point on the contour, the curl at point $\bar{r}$ will be **negative**.

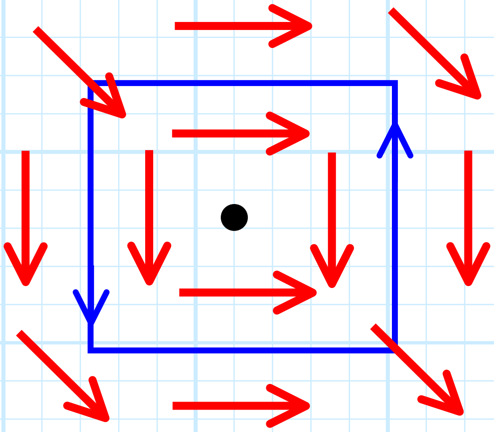


$B_i > 0$

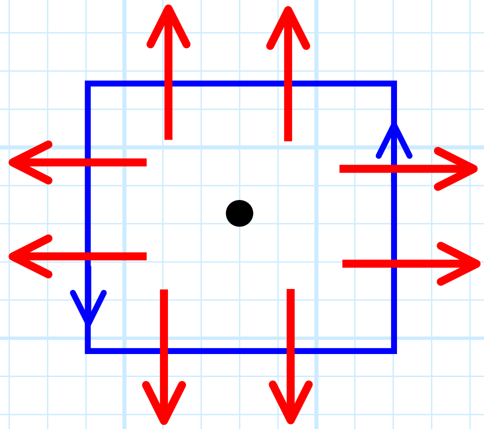


$B_i < 0$

Likewise, **these** vector fields will result in a curl with **zero** value at point $\bar{r}$:



$B_i = 0$

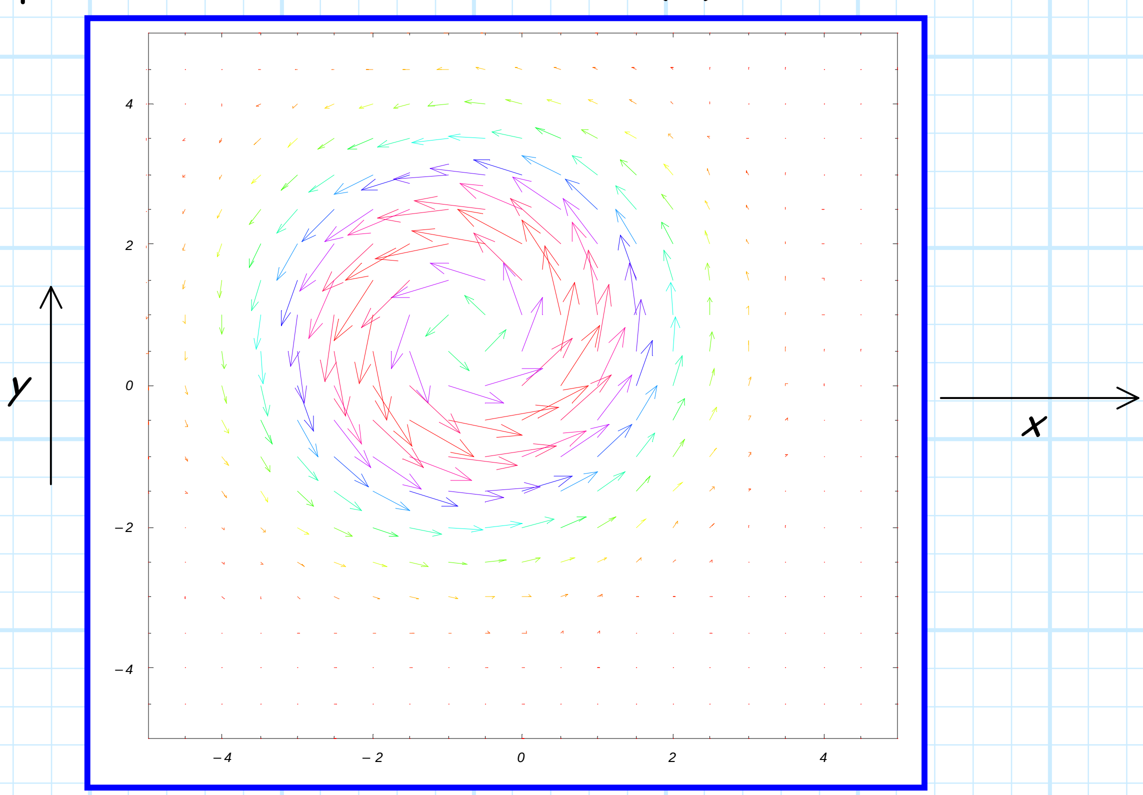


$B_i = 0$

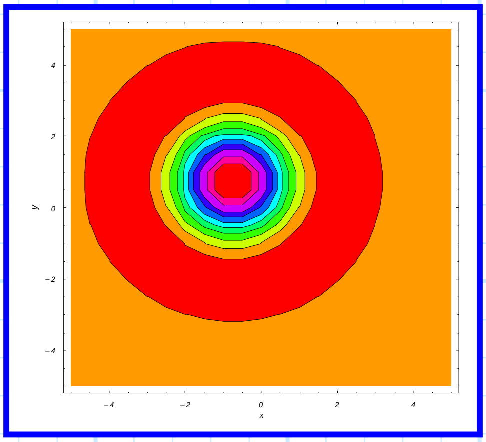
* **Generally**, the curl of a vector field result is in another vector field whose magnitude is positive in some regions of space, negative in other regions, and zero elsewhere.

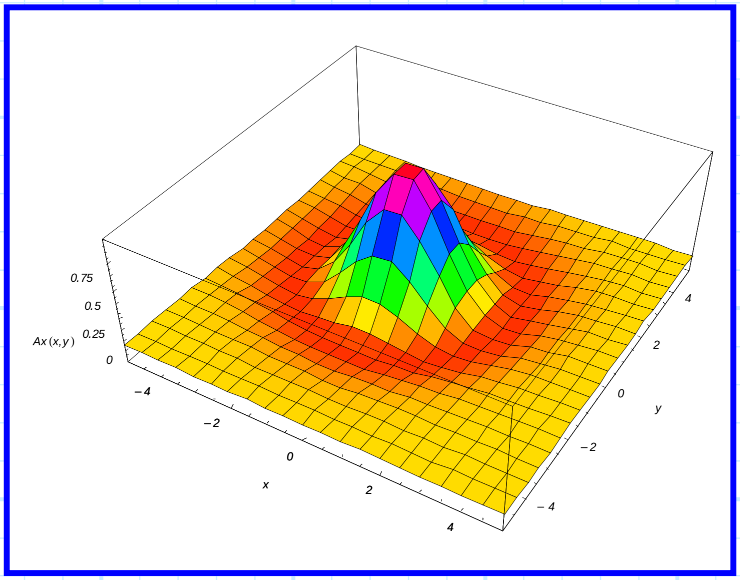
* For most **physical** problems, the curl of a vector field provides another vector field that indicates **rotational sources** (i.e., "paddle wheels") of the original vector field.

For example, consider this vector field $\mathbf{A}(\bar{r})$:

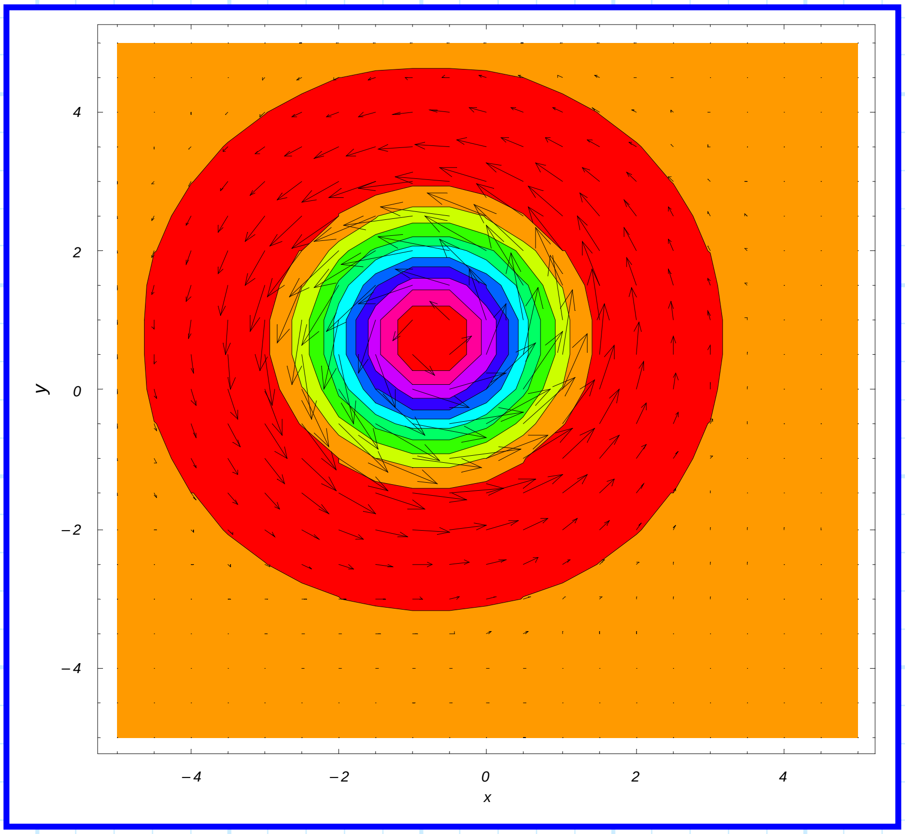


If we take the curl of $\mathbf{A}(\bar{r})$, we get a **vector field** which points in the direction $\hat{a}_z$ at **all** points (x,y). The **scalar component** of this resulting vector field (i.e., $B_z(\bar{r})$) is:



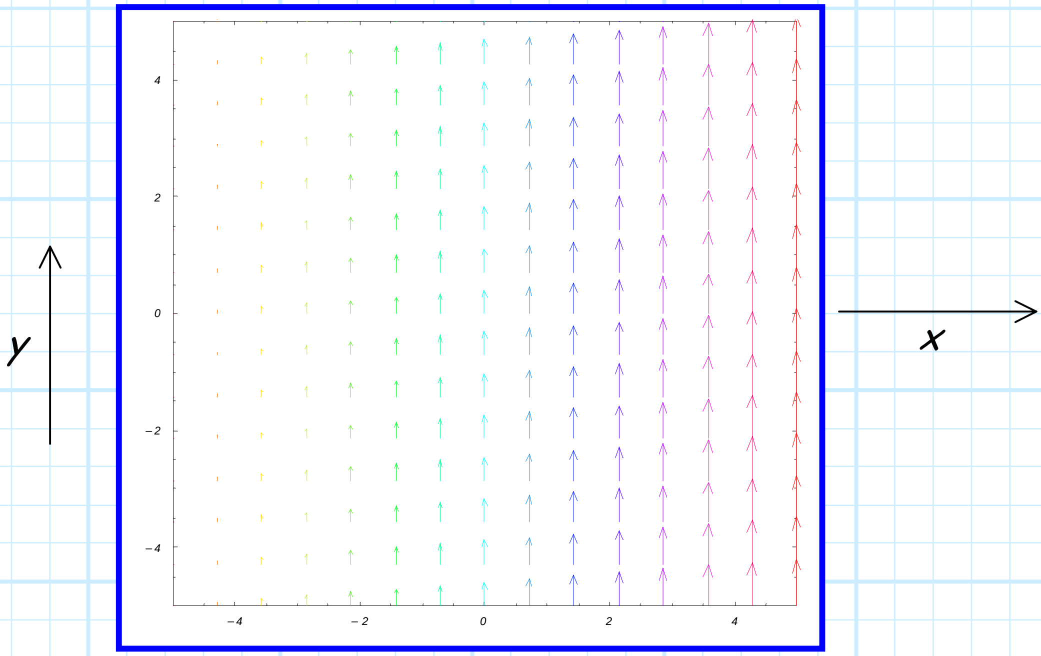


The relationship between the original vector field $\mathbf{A}(\bar{r})$ and its resulting curl perhaps is best shown when plotting both together:

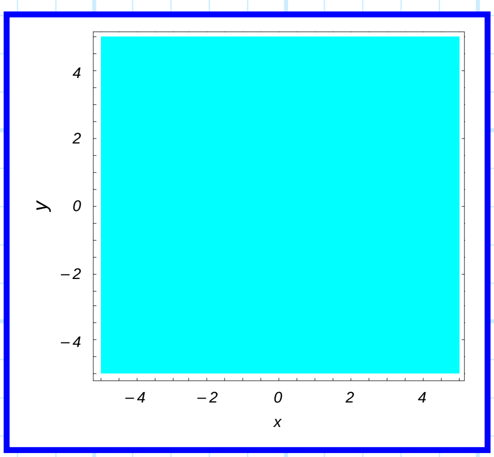


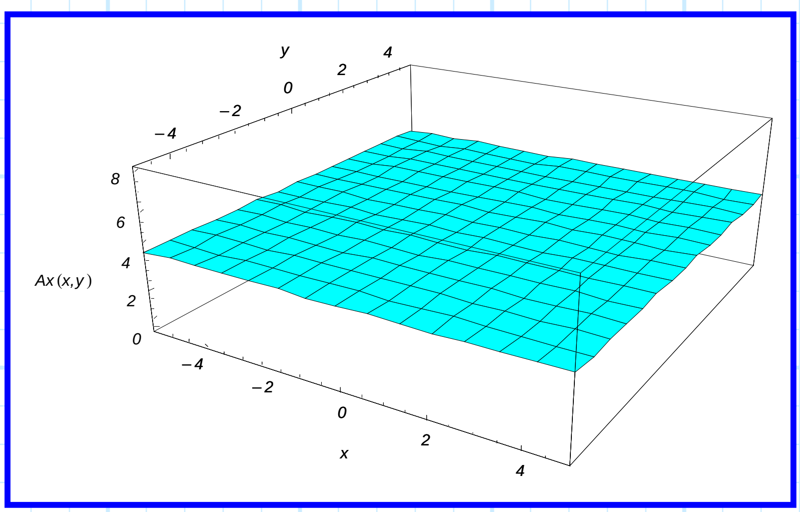
Note this **scalar** component is **largest** in the region near point x=-1, y=1, indicating a "**rotational source**" in this region. This is likewise apparent from the original plot of vector field $\mathbf{A}(\bar{r})$.

Consider now another **vector field**:

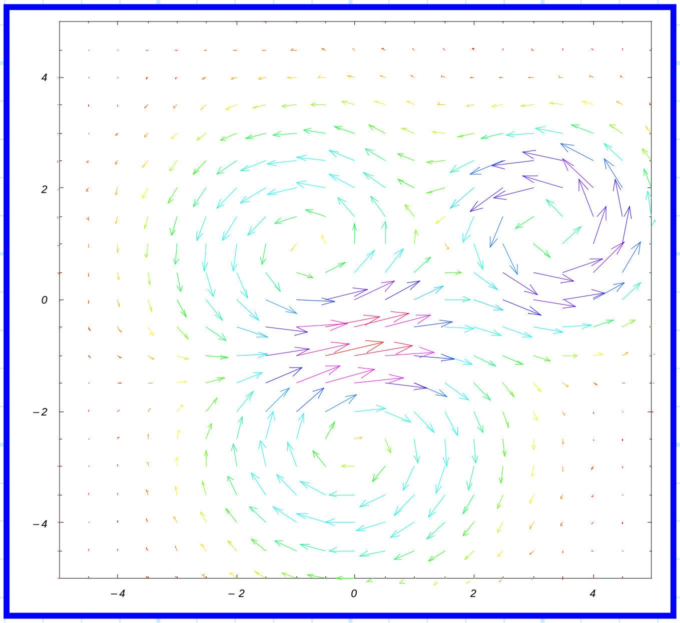


Although at first this vector field **appears** to exhibit no rotation, it in fact has a **non-zero** curl at **every** point ($\mathbf{B}(\bar{r}) = 4.0\ \hat{a}_z$) ! Again, the direction of the resulting field is in the direction $\hat{a}_z$. We plot therefore the **scalar** component in this direction (i.e., $B_z(\bar{r})$):

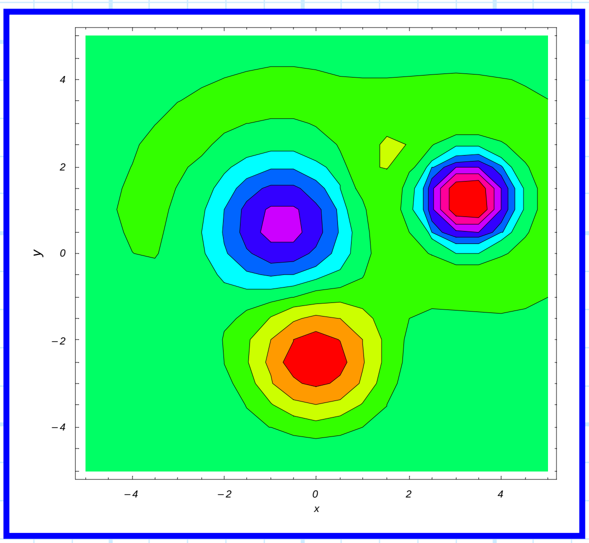


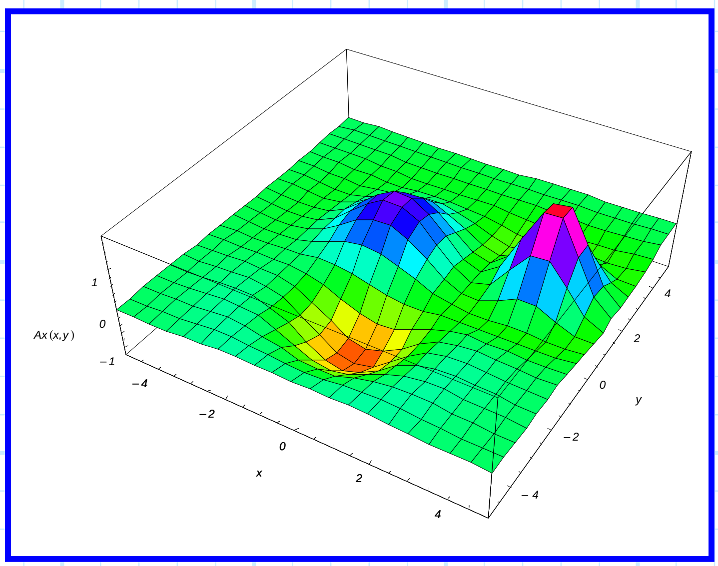


We might encounter a more **complex** vector field, such as:

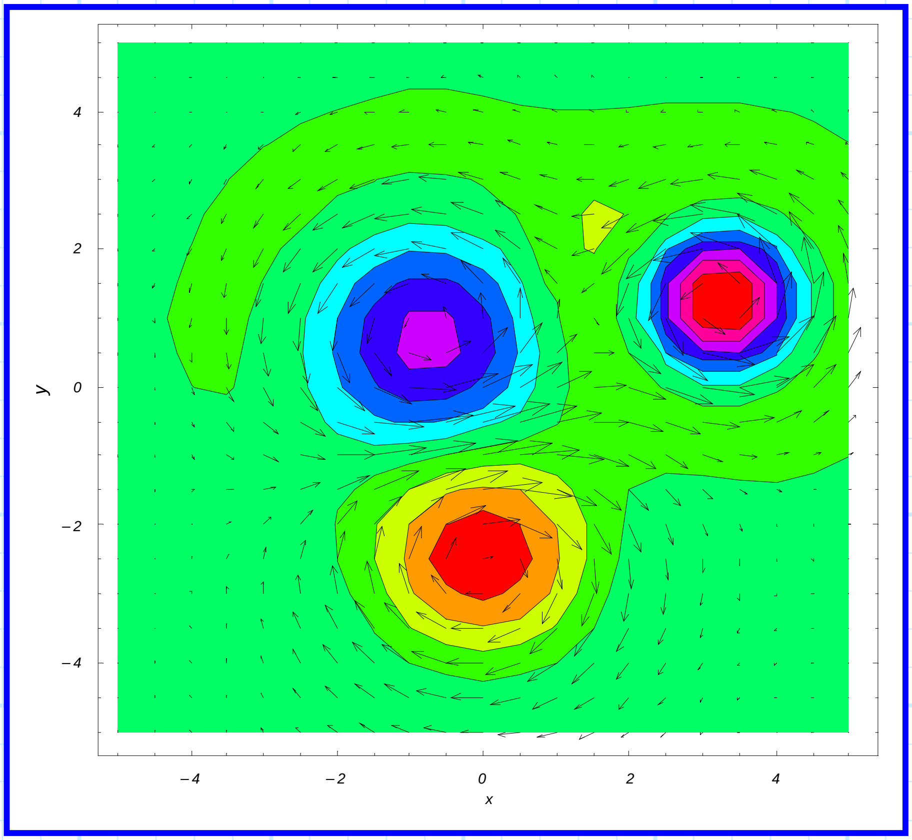


If we take the **curl** of this vector field, the resulting vector field will **again** point in the direction $\hat{a}_z$ at every point (i.e., $B_x(\bar{r}) = B_y(\bar{r}) = 0$). Plotting therefore the scalar component of the resulting vector field (i.e., $B_z(\bar{r})$), we get:

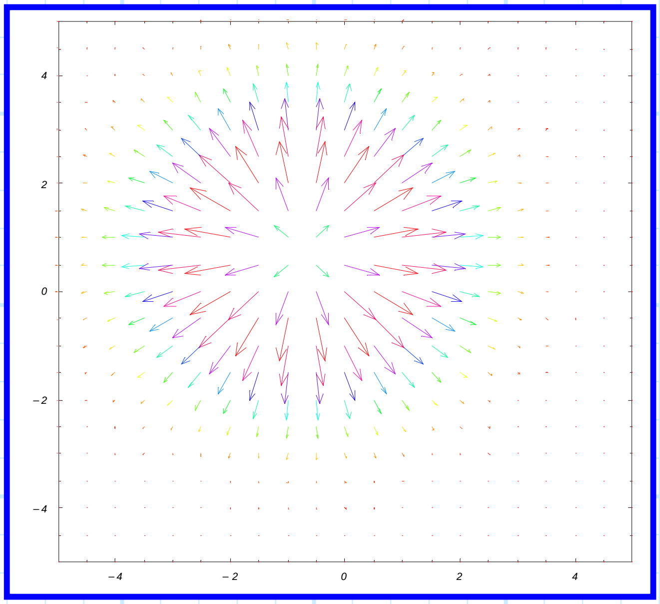


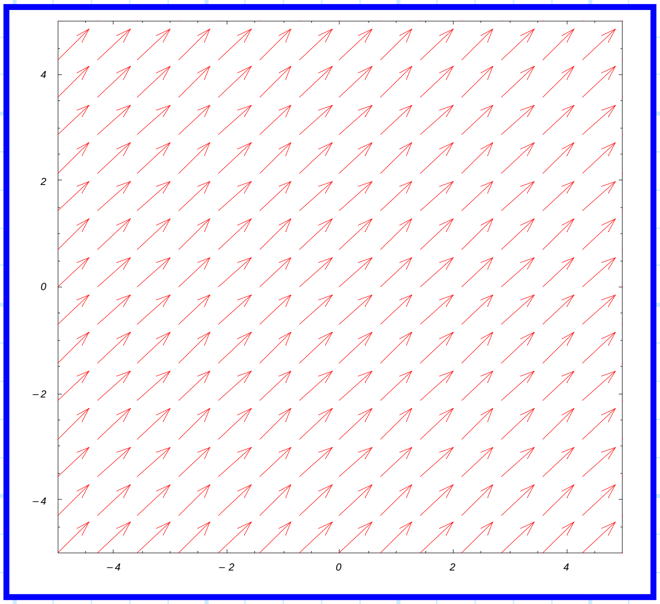


Note these plots indicate that there are **two** regions of large **counter** clockwise rotation in the original vector field, and **one** region of large **clockwise** rotation.



Finally, consider **these** vector fields:





The curl of these vector fields is **zero** at all points. It is apparent that there is no **rotation** in either of these vector fields!